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Large-time Behaviour of Solutions to Phase-Separation Models in One-Dimensional case

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1. Introduction

Let us consider a one-dimensional model for phase separation, which is described as the following system, noted by (P):

$$\frac{u_t}{u^2} + ww_t - u_{xx} = f \quad \text{in } Q := (0, +\infty) \times (-1, 1), \tag{1.1}$$

$$w_t - \{-\kappa w_{xx} + \xi + w^3 - (1+u)w\}_{xx} = 0 \quad \text{in } Q, \tag{1.2}$$

$$\xi \in \partial I_{[-0.5,0.5]}(w) \quad \text{in } Q, \tag{1.3}$$

$$\pm u_x(t, \pm 1) + u(t, \pm 1) = 0 \quad \text{for } t > 0, \tag{1.4}$$

$$w_x(t, \pm 1) \quad \text{for } t > 0, \tag{1.5}$$

$$[-\kappa w_{xx}(t, \cdot\,) + (w(t, \cdot\,))^3 - (1 + u(t, \cdot\,))w(t, \cdot\,)]_x|_{x=\pm 1} = 0 \quad \text{for } t > 0, \tag{1.6}$$

$$u(0, x) = u_0(x), \quad w(0, x) = w_0(x) \quad \text{for } x \in (-1, 1). \tag{1.7}$$

Here, κ is a positive constant; $\partial I_{[-0.5,0.5]}$ is the subdifferential of the indicator function $I_{[-0.5,0.5]}$ of the interval $[-0.5, 0.5]$; f, $h_{\pm}$, u_0 and w_0 are given data .

This system arises in the phase separation of a binary mixture with components A and B.

In this paper, $\theta := -\dfrac{1}{u}$ represents the absolute temperature and w_A the order parameter which is the local concentration of the component A; you note that $-0.5 \le w(t,x) := w_A(t,x) - 0.5 \le 0.5$, and $w(t,x) = 0.5$ (resp. $w(t,x) = -0.5$) means that the physical situation of the system at (t,x) is of pure A (resp. pure B), while $-0.5 < w(t,x) < 0.5$ means that the physical situation is mixture.

About this problem, by N. Kenmochi & M. Niezgódka [6] and [7], we know that (P) has a global and unique solution and under some assumptions on the convergences of the data

$f(t) \longrightarrow 0$ and $h_\pm(t) \longrightarrow h^\infty$ as $t \to +\infty$ in some senses, $u(t) \longrightarrow u^\infty(= h^\infty)$ as $t \to +\infty$ and any ω-limit function w^∞ of the order parameter $w(t)$ is a solution of the following steady-state problem, noted by $(\mathrm{P})^\infty$:

$$-\kappa w^\infty_{xx} + \xi^\infty + (w^\infty)^3 - (1+u^\infty)w^\infty = \sigma \quad \text{in } (-1,1), \tag{1.8}$$

$$\xi^\infty \in \partial I_{[-0.5,0.5]}(w^\infty) \quad \text{in } (-1,1), \tag{1.9}$$

$$\xi^\infty \in L^2(-1,1), \tag{1.10}$$

$$w^\infty_x(\pm 1) = 0, \tag{1.11}$$

$$\frac{1}{2}\int_{-1}^{1} w^\infty(x)dx = m_0, \tag{1.12}$$

where $m_0 = \dfrac{1}{2}\displaystyle\int_{-1}^{1} w_0(x)dx$.
Here, from (1.8) and (1.10), we note that

$$\sigma = \frac{1}{2}\int_{-1}^{1}\{\xi^\infty + (w^\infty(x))^3 - (1+u^\infty)w^\infty(x)\}dx.$$

In this paper, we consider the structure of the ω-limit set of the order parameter w, which is defined by

$$\omega(u_0, w_0) := \{z \in H^1(-1,1);\ w(t_n) \to z \text{ in } H^1(-1,1) \text{ for some } t_n \uparrow +\infty \text{ as } n \to +\infty\}.$$

Notations. For simplicity, we use the following notations:

$H^1(-1,1)$: the usual Sobolev space with norm $|\cdot|_{H^1(-1,1)}$ given by

$$|z|_{H^1(-1,1)} := (|z_x|_{L^2(-1,1)} + |z(-1)|^2 + |z(1)|^2)^{\frac{1}{2}};$$

$H^1(-1,1)^*$: the dual space of $H^1(-1,1)$;

$(\cdot\,,\,\cdot)$: the standard inner product in $L^2(-1,1)$;

$\langle\cdot\,,\,\cdot\rangle$: the duality pairing between $H^1(-1,1)^*$ and $H^1(-1,1)$;

$$a(v,z) := \int_{-1}^{1} v_x(x)z_x(x)dx \quad \text{for } v,\ z \in H^1(-1,1).$$

2. Assumptions and known results

Problems (P) and $(\mathrm{P})^\infty$ are discussed under the following assumptions:

(A1) κ is a positive constant.

(A2) $f \in W^{1,2}_{loc}(0,+\infty; L^2(-1,1)) \cap L^2(0,+\infty; L^2(-1,1))$ such that

$$\sup_{t\geq 0} |f|_{W^{1,2}(t,t+1;L^2(-1,1))} < +\infty.$$

(A3) $h_\pm \in W^{1,2}_{loc}(0,+\infty)$ such that

$$\sup_{t\geq 0}\{|h_+|_{W^{1,2}(t,t+1)} + |h_-|_{W^{1,2}(t,t+1)}\} < +\infty,$$

and for some constant $h^\infty \in (-\infty, 0)$

$$h_\pm - h^\infty \in L^2(0,+\infty).$$

(A4) $h_\pm(t) \in (-\infty, 0]$ for all $t \geq 0$ and there exist positive constants A_1 and A_2 such that

$$\frac{h_\pm(t)}{r} - 1 \geq -A_1|r| - A_2 \quad \text{for all } r \in (-\infty, 0) \text{ and all } t \geq 0.$$

(A5) $u_0 \in H^1(-1,1)$ and $w_0 \in H^1(-1,1)$ such that

$$-\frac{1}{u_0} \in L^2(-1,1),$$

$$w_{0x}(\pm 1) = 0, \quad -0.5 \leq w_0 \leq 0.5 \text{ on } [-1,1]$$

$$-0.5 < m_0 := \frac{1}{2}\int_{-1}^{1} w_0(x)dx < 0.5$$

and there exists $\xi_0 \in L^2(-1,1)$ satisfying

$$\xi_0 \in \partial I_{[-0.5,0.5]}(w_0) \text{ a.e.in } (-1,1), \quad -\kappa w_{0xx} + \xi_0 \in H^1(-1,1).$$

Next, we give a weak variational formulation for (P).

Definition 2.1. For $0 < T < +\infty$ a coupled $\{u, w\}$ of functions $u : [0,T] \longrightarrow H^1(-1,1)$ and $w : [0,T] \longrightarrow H^1(-1,1)$ is called a (weak) solution of (P) on $[0,T]$, if the following conditions (w1)-(w4) are fulfilled:

(w1) $u \in L^\infty(0,T; H^1(-1,1))$,
$-\frac{1}{u}$ is weakly continuous from $[0,T]$ into $L^2(-1,1)$ with

$$\frac{u_t}{u^2} \in L^1(0,T; H^1(-1,1)^*),$$

$w \in L^\infty(0,T; H^1(-1,1)) \cap L^2(0,T; H^2(-1,1))$, $\quad w_t \in L^2(0,T; H^1(-1,1)^*)$,
$ww_t \in L^1(0,T; H^1(-1,1)^*)$.

(w2) $u(0) = u_0$ and $w(0) = w_0$.

(w3) (1.1) holds in the standard variational sense, that is,

$$\frac{d}{dt}\left(-\frac{1}{u(t)} + \frac{1}{2}w^2(t), z\right) + a(u(t), z)$$

$$+(u(t,-1) - h_-(t))z(-1) + (u(t,1) - h_+(t))z(1) = (f(t), z) \tag{2.1}$$

for a.e. $t \in [0,T]$ and all $z \in H^1(-1,1)$.

(w4) For a.e. $t \in [0,T]$,

$$w_x(t, \pm 1) = 0,$$

and there exists a function $\xi \in L^2(0,T;L^2(-1,1))$ such that

$$\xi \in \partial I_{[-0.5,0.5]}(w) \quad \text{for a.e. in } (0,T) \times (-1,1) \tag{2.2}$$

and

$$\frac{d}{dt}(w(t),\eta) + \kappa(w_{xx}(t),\eta_{xx}) - (\xi(t) + (w(t))^3 - (1+u(t))w(t),\eta_{xx}) = 0 \tag{2.3}$$

for all $\eta \in H^2(-1,1)$ with $\eta_x(\pm 1) = 0$ and a.e. $t \in [0,T]$.

As is easily seen from the above definition, for any solution $\{u,w\}$ of (P) on $[0,T]$ it holds that

$$\frac{1}{2}\int_{-1}^{1} w(t,x)dx = \frac{1}{2}\int_{-1}^{1} w_0(x)dx = m_0$$

and

$$\frac{u_t}{u^2} + ww_t \in L^\infty(0,T;H^1(-1,1)^*).$$

Also, the inequalities $-0.5 \le m_0 \le 0.5$ are necessary in order for (P) to have a solution; if $m_0 = 0.5$ (resp. -0.5), then we see that $w \equiv 0.5$ (resp. -0.5).
We say that a couple $\{u,w\}$ of functions $u : [0,+\infty) \longrightarrow H^1(-1,1)$ and $w : [0,+\infty) \longrightarrow H^1(-1,1)$ is a solution of (P) on $[0,+\infty)$, if it is a solution of (P) on $[0,T]$ for every finite $T > 0$.

We now recall an existence and uniqueness results.

Theorem 2.1. *[cf. 7] Assume that (A1)-(A5) hold. Then (P) has one and only one solution $\{u,w\}$ on $[0,+\infty)$, and it satisfies that for every finite $T > 0$*

$$\begin{cases} u \in L^2(0,T;H^2(-1,1)), \quad u_t \in L^2(0,T;L^2(-1,1)), \\ w \in L^\infty(0,T;H^2(-1,1)), \quad w_t \in L^\infty(0,T;H^1(-1,1)^*) \cap L^2(0,T;H^1(-1,1)), \\ \xi \in L^\infty(0,T;L^2(-1,1)), \end{cases} \tag{2.4}$$

where ξ is the function as in (w4) of Definition 2.1.

As to global estimates for solutions we have the following theorem

Theorem 2.2. *[cf. 3] Assume that (A1)-(A5) hold. Let $\{u,w\}$ be the solution of (P) on $[0,+\infty)$. Then,*

$$u - u^\infty \in L^2(0,+\infty;H^1(-1,1)), \quad u \in L^\infty(0,+\infty;H^1(-1,1)), \tag{2.5}$$

$$\sup_{t \ge 0} |u_t|_{L^2(t,t+1;L^2(-1,1))} < +\infty, \tag{2.6}$$

$$w \in L^\infty(0,+\infty;H^2(-1,1)), \tag{2.7}$$

$$w_t \in L^\infty(0,+\infty;H^1(-1,1)) \cap L^2(0,+\infty;H^1(-1,1)^*) \tag{2.8}$$

and

$$\sup_{t\geq 0} |w_t|_{L^2(t,t+1;H^1(-1,1))} < +\infty. \tag{2.9}$$

From this theorem, we have the following corollary.

Corollary 2.1. *[cf. 3] Under the same assumptions as in Theorem 2.2, the following statements hold:*

(a) $u(t) \longrightarrow u^\infty(= h^\infty)$ *weakly in* $H^1(-1,1)$ *as* $t \to +\infty$.

(b) *The* ω*-limit set* $\omega(u_0, w_0)$ *is non-empty, compact and connected in* $H^1(-1,1)$*. Also,* $\omega(u_0, w_0)$ *is bounded in* $H^2(-1,1)$.

(c) $\lim_{t\to+\infty} \left\{ \frac{\kappa}{2}|w_x(t)|^2_{L^2(-1,1)} + \int_{-1}^{1} \left(\frac{1}{4}(w(t,x))^4 - \frac{1}{2}(1+u^\infty)(w(t,x))^2 \right) dx \right\}$ *exists.*

(d) *Any* ω*-limit function* $v \in \omega(u_0, w_0)$ *is solution of (P)*$^\infty$.

From this corollary, the absolute temperature $-\frac{1}{u(t)}$ converges to a constant $-\frac{1}{u^\infty}$. On the other hand, in general the order parameter $w(t)$ does not converge, but any ω-limit function of $w(t)$ is a solution of (P). So, in the next section we consider the structure of the solutions of (P)$^\infty$ and ω -limit set $\omega(u_0, w_0)$.

3. The structure of ω-limit set $\omega(u_0, w_0)$

In this section, we consider the structure of the solution of (P)$^\infty$ and $\omega(u_0, w_0)$. Here, we note that the shape of the function $w^3 - (1+u^\infty)w$ chages as u^∞ changes. From this results and (a) of Corollary 2.1, we consider $u^\infty(= h^\infty)$ as a controll parameter.
For simplicity, we use the following notations:

$$G(w; u^\infty) := \int_0^w \{v^3 - (1+u^\infty)v\} dv$$

and

$$H(w; \sigma, u^\infty) := \int_0^w \{v^3 - (1+u^\infty)v - \sigma\} dv = G(w; u^\infty) - \sigma w.$$

Lemma 3.1. *[cf. 3] Let* w^∞ *be any solution of (P)*$^\infty$ *and put* $b = H(w^\infty(-1); \sigma, u^\infty)$*. Then,* $H(w^\infty(x); \sigma, u^\infty) \geq b$ *for all* $x \in [-1,1]$.
Moreover, $w^\infty_x(x) = 0$ *if and only if* $H(w^\infty(x); \sigma, u^\infty) = b$*, hence* $H(w^\infty(1); \sigma, u^\infty) = b$.
Proof. Multiplying (1.8) by w^∞_x and integrating it over $[-1, x]$, from (1.11) we have

$$-\frac{\kappa}{2}|w^\infty_x(x)|^2 + H(w^\infty(x); \sigma, u^\infty) = b \quad \text{for all } x \in [-1,1].$$

Hence, this lemma holds. $\diamondsuit$.

Next, since there exist two cases of the shape of the function $w^3 - (1+u^\infty)w$, we consider the two cases one by one.

(i) Case 1: $u^\infty \leq -1$

In this case, $w^3 - (1+u^\infty)w$ is strictly increasing. So, there exists one and only one solution $\zeta(\sigma)$ of the algebraic equation $w^3 - (1+u^\infty)w = \sigma$, that is, $H(w;\sigma,u^\infty)$ has the following properties:

$$H(w;\sigma,u^\infty) \text{ is strictly decreasing on } (-\infty,\zeta(\sigma)),$$

$$H(w;\sigma,u^\infty) \text{ is strictly increasing on } (\zeta(\sigma),+\infty)$$

and

$$H(w;\sigma,u^\infty) \geq H(\zeta(\sigma);\sigma,u^\infty).$$

Theorem 3.1. *(P)$^\infty$ has no non-constant solution.*

Proof. We assume that w^∞ is a non-constant solution of (P)$^\infty$.

Then, from Lemma 3.1 and the properties of $H(w;\sigma,u^\infty)$ we can see that there exist two following cases (α) and (β) for w^∞.

(α) $w^\infty(-1) \leq \zeta(\sigma)$ and w^∞ is decreasing on $[-1,1]$.

(β) $w^\infty(-1) \geq \zeta(\sigma)$ and w^∞ is increasing on $[-1,1]$.

In both cases (α) and (β) we have $w_x^\infty \neq 0$ on $(-1,1]$ which contradicts the boundary condition $w_x^\infty(1) = 0$. Therefore, we obtain this theorem. $\diamondsuit$

From Theorem 3.1, we can see that the following theorem, easily.

Theorem 3.2. *(P)$^\infty$ has a constant solution $v \equiv m_0$ on $[-1,1]$, only.*
Moreover, $\sigma = G(m_0;u^\infty)$ and $b = (1-m_0)G(m_0;u^\infty)$.

Proof. From (1.12), $w^\infty \equiv m_0$ on $[-1,1]$ must hold. Since $-0.5 < m_0 < 0.5$, $\xi^\infty \equiv 0$ on $[-1,1]$. So,

$$\begin{aligned} \sigma &= \frac{1}{2}\int_{-1}^{1}\{\xi^\infty + m_0^3 - (1+u^\infty)m_0\}dx \\ &= m_0^3 - (1+u^\infty)m_0 = G(m_0;u^\infty)0. \end{aligned}$$

Moreover,

$$b = G(m_0;u^\infty) - \sigma m_0 = (1-m_0)G(m_0;u^\infty). \qquad \diamondsuit.$$

Remark 3.1. From Corollary 2.1 and 3.2, the order parameter $w(t)$ converges $w^\infty \equiv m_0$ as $t \to +\infty$. So, there exists one and only one ω-limit set $\omega(u_0,w_0) = \{w^\infty\}$.

Case 2: $-1 < u^\infty < 0$

In this case, $w^3 - (1+u^\infty)w$ is non-monotone and N-shape. So, we consider the case

when $m_0 = 0$.
Here, We note that there exist two cases for the position of constraints -0.5 and 0.5.

First case, when $-0.75 \leq u^\infty < 0$, these constraints are outside of zero points of $w^3 - (1+u^\infty)w$, that is,

$$-0.5 \leq -\sqrt{1+u^\infty} < 0 < \sqrt{1+u^\infty} \leq 0.5.$$

Second case, when $-1 < u^\infty < -0.75$, they are inside, that is,

$$-\sqrt{1+u^\infty} < -0.5 < 0 < 0.5 < \sqrt{1+u^\infty}.$$

At first, by using the same technique as in Theorem 3.2, we obtain the following theorem about a constant solution.

Theorem 3.3. *(P)$^\infty$ has one and only one constant solution $w^\infty \equiv 0$ on $[-1,1]$. Moreover, in this case $\sigma = b = 0$.*

In the rest of this case, we consider non-constant solutions of (P)$^\infty$. To do so, we note that there exist three following cases of the shape of the function $H(w;\sigma,u^\infty)$ by the value of σ.

(a) When $\sigma \geq 2\left(\dfrac{1+u^\infty}{3}\right)^{\frac{3}{2}}$, $H(w;\sigma,u^\infty)$ has the following properties:

$$H(w;\sigma,u^\infty) \text{ is strictly decreasing on } (-\infty, \zeta_+(\sigma)),$$

$$H(w;\sigma,u^\infty) \text{ is strictly increasing on } (\zeta_+(\sigma), +\infty)$$

and

$$H(w;\sigma,u^\infty) \geq H(\zeta_+(\sigma);\sigma,u^\infty),$$

where $\zeta_+(\sigma)$ is a root of the algebraic equation $w^3 - (1+u^\infty)w = \sigma$ such that $\zeta_+(\sigma) > -\left(\dfrac{1+u^\infty}{3}\right)^{\frac{1}{2}}$.

(b) When $\sigma \leq -2\left(\dfrac{1+u^\infty}{3}\right)^{\frac{3}{2}}$, $H(w;\sigma,u^\infty)$ has the following properties:

$$H(w;\sigma,u^\infty) \text{ is strictly decreasing on } (-\infty, \zeta_-(\sigma)),$$

$$H(w;\sigma,u^\infty) \text{ is strictly increasing on } (\zeta_-(\sigma), +\infty)$$

and

$$H(w;\sigma,u^\infty) \geq H(\zeta_-(\sigma);\sigma,u^\infty),$$

where $\zeta_-(\sigma)$ is a root of the algebraic equation $w^3 - (1+u^\infty)w = \sigma$ such that $\zeta_-(\sigma) < \left(\dfrac{1+u^\infty}{3}\right)^{\frac{1}{2}}$.

(c) When $-2\Big(\dfrac{1+u^\infty}{3}\Big)^{\frac{3}{2}} < \sigma < 2\Big(\dfrac{1+u^\infty}{3}\Big)^{\frac{3}{2}}$, $H(w;\sigma,u^\infty)$ has the following properties:

$$H(w;\sigma,u^\infty) \text{ is strictly decreasing on } (-\infty,\zeta_-(\sigma)) \cup (\zeta(\sigma),\zeta_+(\sigma))$$

and

$$H(w;\sigma,u^\infty) \text{ is strictly increasing on } (\zeta_-(\sigma),\zeta(\sigma)) \cup (\zeta_+(\sigma),+\infty),$$

where $\zeta_-(\sigma)$, $\zeta(\sigma)$ and $\zeta_+(\sigma)$ are roots of the algebraic equation $w^3-(1+u^\infty)w=\sigma$ such that $\zeta_-(\sigma) < -\left(\dfrac{1+u^\infty}{3}\right)^{\frac{1}{2}} < \zeta(\sigma) < \left(\dfrac{1+u^\infty}{3}\right)^{\frac{1}{2}} < \zeta_+(sigma)$.

To the cases (a) and (b), by using the same technique as in Theorem 3.1, we can see that the following theorem holds.

Theorem 3.4. *We assume that* $\sigma \le -2\Big(\dfrac{1+u^\infty}{3}\Big)^{\frac{3}{2}}$ *or* $\sigma \ge 2\Big(\dfrac{1+u^\infty}{3}\Big)^{\frac{3}{2}}$. *Then,* $(P)^\infty$ *has no non-constant solution.*

From this theorem, we only consider the case (c). In this case, by the results of A. Ito & N. Kenmochi [6], we know that the following theorem holds.

Theorem 3.5. *Let* w^∞ *be non-constant solution of* $(P)^\infty$. *Then,*

(1) $\sigma = 0$.

(2) *If* $-0.75 \le u^\infty < 0$, *then all* ω*-limit set* $\omega(u_0,w_0)$ *is a singleton, that is,* $\omega(u_0,w_0) = \{w^\infty\}$. *Moreover, the number of* $\omega(u_0,w_0)$ *is equal to* $2n_1+1$, *where* n_1 *is the number of* b *with* $G(-\sqrt{1+u^\infty};u^\infty) = G(\sqrt{1+u^\infty};u^\infty) < b < 0$ *satisfying the following condition (*):*

(*) There exist a natural number $N(b)$ *such that* $N(b)I(b)=2$,

where $\pm\eta(b)$ *are roots of the algebraic equation* $G(w;u^\infty)=b$ *such that* $-\sqrt{1+u^\infty} < -\eta(b) < 0 < \eta(b) < \sqrt{1+u^\infty}$ *and*

$$I(b) := \left(\frac{\kappa}{2}\right)^{\frac{1}{2}} \int_{-\eta(b)}^{\eta(b)} \frac{1}{\{G(w;u^\infty)-b\}} dw.$$

(3) *If* $-1 < u^\infty < -0.75$, *there exist two posibilities (i) and (ii) of the structure of* $\omega(u_0,w_0)$:

(i) $\omega(u_0,w_0)$ *is a singleton.*

(ii) $\omega(u_0,w_0)$ *contains a continuum of the solutions of* $(P)^\infty$. *Moreover, in this case the following properties hold:*

(α) $b = G(-0.5;u^\infty) = G(0.5;u^\infty)$.

(β) $\eta(b) = 0.5$. *Hence, in particular boundary values* $w^\infty(-1)$ *and* $w^\infty(1)$ *take* -0.5 *or* 0.5.

(γ) $|J_B| = |J_A|$, *where* $|J_A|$ *and* $|J_B|$ *are the length of the pure region of the components A and B, respectively.*

Moreover, the nunber of $\omega(u_0, w_0)$ *is equal to* $2n_1 + 2n_2 + 1$, *where* n_1 *is the number of* b *with* $G(-0.5; u^\infty) = G(0.5; u^\infty) < b < 0$ *satisfying (*) and* n_2 *is the number of the natural number* n *satisfying the following conditions (**):*

$$(**)\ nI(G(-0.5; u^\infty)) = nI(G(0.5; u^\infty)) \leq 2.$$

From this theorem, we are interested in the case when (ii) of (3).
But, this case is very dependent upon the coefficient κ.
At last, we give the theorem to show that ω-limit set is very dependent upon κ.

Theorem 3.6. *If* κ *is large enough to satisfy the following codition (**)*

$$2I(G(0.5; u^\infty)) \geq 2.$$

Then, all ω*-limit set are singleton, that is, the order parameter* $w(t)$ *converges to some* ω*-limit function* w^∞ *as* $t \to +\infty$.

Proof. It is clear from the above theorem. $\diamondsuit$.

Remark 3.2. We can see that ω-limit set is very dependent upon the length of the interval when κ is fixed.

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